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On the 1-Factors of n-Connected Graphs*

JOSEPH ZAKS

University of Washington, Seattle, Washington 98195,
and Wayne State University, Detroit, Michigan 48202

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L. W. Beineke and M. D. Plummer have recently proved [1] that every n-connected graph with a 1-factor has at least n different 1-factors. The main purpose of this paper is to prove that every n-connected graph with a 1-factor has at least as many as $n(n-2)(n-4)\ldots 4\cdot 2$, (or: $n(n-2)(n-4)\ldots 5\cdot 3$) 1-factors. The main lemma used is: if a 2-connected graph G has a 1-factor, then G contains a vertex V (and even two such vertices), such that each edge of G, incident to V, belongs to some 1-factor of G.

I. INTRODUCTION AND STATEMENT OF RESULTS. We consider here finite graphs, without loops or double edges. A *1-factor* of a graph G is a subgraph of G, which contains all the vertices of G, each with valence 1.

Let $f(n)$ denote the maximum k for which the following is true: "If G is an n-connected graph with a 1-factor then G has at least k (different) 1-factors."

One can now reformulate Theorem 1 of [1] simply as $f(2)\geqslant 2$, and Theorem 2 of [1] becomes $f(n)\geqslant n$, for all $n\geqslant 2$.

Our extension of [1] is the following

THEOREM 2. *$f(n)\geqslant n(n-2)(n-4)\ldots 4\cdot 2$ for even n, while for odd n, $f(n)=n(n-2)(n-4)\ldots 5\cdot 3$.*

Let $F(G)$ denote the subgraph of G, which is the union of all the 1-factors of G. A vertex V of G will be called *totally covered by $F(G)$* if all the edges of G, incident to V, belong to $F(G)$.

The following is our main idea:

LEMMA 4. *If G is a 2-connected graph with a 1-factor, then G contains a vertex V which is totally covered by $F(G)$.*

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To the proof of this lemma we have the following

COROLLARY 1. *If G is a 2-connected graph with a 1-factor, then G contains two vertices which are totally covered by $F(G)$.*

A simpler proof of Theorem 1 of [1] is given here, due to B. Grünbaum.

II. TERMINOLOGY. A graph G is *connected* if for each pair of its vertices V and W, G contains a $V - W$ path P: an ordered collection $(V_1, E_1, V_2, E_2, \ldots, E_k, V_{k+1})$ of vertices V_i and edges E_i of G, such that $V_1 = V$, $V_{k+1} = W$ and for all $1 \leqslant i \leqslant k$, E_i is the edge (V_i, V_{i+1}) of G.

A graph G is *n-connected between two of its vertices V and W* if G contains n $V - W$ paths $P_1, \ldots, P_n$ such that $P_i \cap P_j = \{V, W\}$ for all $i \neq j$, $1 \leqslant i$, $j \leqslant n$. G is *n-connected* if it is n-connected between each pair of its vertices, and it contains at least two vertices.

It is well known that G is n-connected if it contains at least $n + 1$ vertices, and the deletion of no set of $n - 1$ or fewer vertices disconnects G.

Clearly, if G is n-connected, having a vertex V and an edge E, then both $G - V$ and $G - E$ are $n - 1$ connected.

The following is a well known property; the proof is added for completeness:

LEMMA 1. *If G is a 2-connected graph, and $E = (V_1, V_2)$ is an edge of G, such that $G - E$ is only 1-connected, then $G - E$ is only 1-connected between V_1 and V_2.*

Proof. Let U and W be two vertices of G (with at most one of them being V_1 or V_2) such that G is 2-connected between U and W, and $G - E$ is only 1-connected between U and W. Let α_1 and α_2 be two disjoint (except for common end points) $U - W$ paths in G. Therefore E belongs to one of α_1 and α_2, say to α_1. $G - E$ is only 1-connected between U and W, therefore $G - E$ has a cut vertex X such that U and W belong to different connected components of $G - \{E, X\}$. Since α_2 connects U and W in $G - E$, $X \in \alpha_2$.

Since $E \in \alpha_1$, U is path-connected to one end point of E, in $G - \{E, X\}$, while W is path-connected to the other end point of E.

If $G - E$ is 2-connected between V_1 and V_2, there are two disjoint (except for common end points) $V_1 - V_2$-paths in $G - E$, say β_1 and β_2.

X can be only on one of β_1 and β_2, therefore V_1 and V_2 are connected in $G - \{E, X\}$, and therefore U and W are connected in $G - \{E, X\}$; this is a contradiction.

Therefore $G - E$ is not 2-connected between V_1 and V_2, as promised by the lemma.

III. THEOREM 1 (Beineke-Plummer). *If a 2-connected graph G has a 1-factor, then G has at least another 1-factor.*

Proof (Grünbaum). The proof is by induction on the number of the edges, the assertion being obvious if G has 4 edges. Assuming the theorem is true for all the graphs with $<k$ edges, let G be a 2-connected graph with k edges and with a 1-factor F. If G has an edge, both end points of which are of valence 2, there is a trivial reduction, of both G and F.

Excluding this possibility, we observe that there necessarily exists an edge $E = (V_1, V_2)$ of $G - F$ both end points of which have valence $\geqslant 3$.

Let $G^* = G - E$. If G^* is 2-connected, then, since F is a 1-factor of G^* (as well as of G), it follows from the inductive assumption that G^* has at least two 1-factors, which are 1-factors of G, as promised. If G^* is not 2-connected, it has a cut vertex V; $G^* - V$ has precisely two connected components, say G_1 and G_2 (with $V_i \in G_i$, $i = 1, 2$), since G is 2-connected.

If F contains the edge (V_1, V), then the graph $G - G_2$, obtained by deleting G_2 from G, is 2-connected and has a 1-factor; hence it has, by the inductive assumption, at least one other 1-factor, which may be combined with $F \cap G_2$ to yield a new 1-factor of G. Similarily if (V_2, V) is an edge of F.

If neither of these cases arises, let us assume the notation is such that the edge of F which contains V has its other end point in G_1. Let H_1 be the graph obtained from $G - G_2$ by adding the edge (V_1, V) (if it is not already there), and let H_2 be the graph obtained from $G - G_1$ by adding the edges (V_1, V_2) and (V_1, V). Clearly, H_1 and H_2 are 2-connected, $F_1 = H_1 \cap F$ is a 1-factor of H_1, while $F_2 = (V_1, V) \cup (H_2 \cap F)$ is a 1-factor of H_2. Therefore there exist 1-factors F_i^* of H_i different from F_i, $i = 1, 2$.

If F_1^* does not contain the edge (V_1, V) then $(H_2 \cap F) \cup F_1^*$ is a 1-factor of G, different from F. If F_2^* does contain the edge (V_1, V), then $[(F_2^* - (V_1, V)] \cup F_1$ is a 1-factor of G different from F. If neither of these cases happens, then $(F_1^* \cup F_2^*) - (V_1, V)$ is the desired 1-factor of G.

This completes the proof of the theorem.

IV. With very little effort we can improve considerably the inequality $f(n) \geqslant n$, of [1], while using $f(2) \geqslant 2$, as follows:

LEMMA 2. *For each $n \geqslant 3, f(n) \geqslant f(n-1) + f(n-2)$.*

Proof. Let G be an n-connected graph with a 1-factor F, $n \geqslant 3$. Since G is 2-connected, it has by [1] another 1-factor, say F_1. Let $E = (V_1, V_2)$ be an edge of G, such that $E \in F$ and $E \notin F_1$.

$G - E$ is $(n-1)$-connected, and has F_1 as a 1-factor, therefore $G - E$ has at least $f(n-1)$ 1-factors, each one of which is a 1-factor of G, which does not contain E. $G^* = G - \{V_1, V_2\}$ is $(n-2)$-connected, and has $F - E$ as a 1-factor, hence G^* has at least $f(n-2)$ 1-factors. Adding the edge E to a 1-factor of G^* yields a 1-factor of G; therefore G has at least $f(n-2)$ 1-factors, each one of which contains the edge E.

Clearly, these two collections of 1-factors of G are disjoint, therefore G has at least $f(n-1) + f(n-2)$ 1-factors.

This completes the proof of the lemma.

V. Main Results. The main ideas of this paper are contained in the following two lemmas:

Lemma 3. *If every n-connected graph with a 1-factor contains a vertex V, such that at least n edges of G, incident to V, belong to $F(G)$, then for each k, $k \geqslant 1$,*

$$f(k) \geqslant k(k-2)(k-4)\ldots .$$

Proof. The proof is by induction on k, as follows: for $k = 1$ this is the triviality that $f(1) \geqslant 1$; for $k = 2$ the claim is that $f(2) \geqslant 2$ which has been proved in [1] and in Section III here. To complete the proof it suffices to show that, for each k, $k \geqslant 3$,

$$f(k) \geqslant k \cdot f(k-2) .$$

Let G be a k-connected graph with a 1-factor. From the assumption of the lemma it follows that G contains vertices $V, V_1, \ldots, V_k$ and edges $E_i = (V, V_i)$, $1 \leqslant i \leqslant k$, such that each E_i belong to some 1-factor F_i of G.

Let $G_i = G - \{V, V_i\}$, for each $1 \leqslant i \leqslant k$. Then each graph G_i is $(k-2)$ connected, and has a 1-factor: $F_i - E_i$. Let $\tilde{F}_i$ be the collection of all the 1-factors of G_i; therefore, card $\tilde{F}_i \geqslant f(k-2)$. Let $F_i^* = \{A \cup E_i \mid A \in \tilde{F}_i\}$, for each $1 \leqslant i \leqslant k$. Clearly, each member of F_i^* is a 1-factor of G. Moreover, $F_i^* \cap F_j^* = \varnothing$ for $i \neq j$, $1 \leqslant i, j \leqslant k$, because each member of F_i^* contains the edge (V, V_i) and each member of F_j^* contains the edge (V, V_j). Therefore G has at least $k \cdot f(k-2)$ 1-factors, and the proof of Lemma 3 is completed.

In order to prove that the condition of Lemma 3 holds, we prefer to prove the following stronger claim:

Lemma 4. *If G is a 2-connected graph with a 1-factor F, then G contains a vertex V which is totally covered by $F(G)$.*

Proof. The proof is by induction on the number k of the edges of G. The induction begins with $k = 4$, in which case $G = F(G)$, therefore all the vertices of G are totally covered by $F(G)$, because in this case G must be a (simple) cycle of length 4.

Let us assume that the assertion is true for all the appropriate graphs with less than k edges, and let G be a 2-connected graph with a 1-factor, having k edges.

To prove that G has a vertex, which is totally covered by $F(G)$, we suppose, on the contrary, that G has no such a vertex.

CLAIM 1. *If G has an edge E, which belongs to one 1-factor F^*, but does not belong to another 1-factor F^{**}, of G, then $G - E$ is only 1-connected.*

Proof. Suppose an edge E of G belongs to F^*, but not to F^{**}, and $G - E$ is 2-connected.

$G - E$ has $k - 1$ edges, and has a 1-factor F^{**}, hence, by the inductive assumption, $G - E$ contains a vertex V, which is totally covered by $F(G - E)$. Now, $E \in F^*$ therefore $E \in F(G)$, and clearly $F(G - E) \subset F(G)$; therefore the vertex V of G is totally covered by $F(G)$; this contradicts our supposition that G contains no such a vertex, and hence completes the proof of Claim 1.

Now, G is 2-connected and has a 1-factor F, therefore, by Theorem 1, G has another 1-factor F'. The subgraph $(F' - F) \cup (F - F')$ of G is not empty, and all of its vertices are of valence 2; therefore its connected components are (simple) cycles of length r, where r is even and $r \geqslant 4$.

Let $S = (V_1, E_1, V_2, E_2, ..., V_r, E_r, V_1)$ be one such component, where $E_{2i} \in F$ and $E_{2i+1} \in F'$, $1 \leqslant 2i, 2i + 1 \leqslant r$.

For each $1 \leqslant i \leqslant r$, we have by Claim 1 that $G - E_i$ is only 1-connected, because each one of the edges E_i belongs to precisely one of the two 1-factors of G, F and F'.

Since no vertex of G is totally covered by $F(G)$, and clearly $S \subset F \cup F' \subset F(G)$, therefore each vertex V_i of S has valence $\geqslant 3$. Therefore there are edges $E_1{}^*, ..., E_r{}^*$ in G which are not edges of S, such that, for all $1 \leqslant i \leqslant r$, V_i is a vertex of $E_i{}^*$. Let $E_i{}^* = (V_i, V_i{}^*)$, for all $1 \leqslant i \leqslant r$.

G is 2-connected, therefore $G - V_i$, for each $1 \leqslant i \leqslant r$, is (at least) 1-connected. Let α_i be a path in $G - V_i$, connecting the vertex $V_i{}^*$ with some vertex V_i^{**} of $S - V_i$, such that V_i^{**} is the only vertex of S on α_i $(1 \leqslant i \leqslant r)$. Clearly $V_i^{**} \neq V_i$, and, if $V_i{}^* \in S$, then α_i is the degenerate path, consisting of the vertex $V_i{}^*$ only, and then $V_i^{**} = V_i{}^*$.

CLAIM 2. *V_i^{**} is not a neighbor, in S, of V_i, for all $1 \leqslant i \leqslant r$. (Remark that $r \geqslant 4$.)*

Proof. Suppose, on the contrary, that, for some i, $1 \leqslant i \leqslant r$, V_i^{**} is a neighbor, in S, of V_i; say $V_i^{**} = V_{i+1}$.[†] G is 2-connected, while $G - E_i$ is only 1-connected, hence by Lemma 1 $G - E_i$ is only 1-connected between V_i and V_{i+1}. However, $\alpha_i \cup E_i^*$ and $S - E_i$ are two disjoint $V_i - V_{i+1}$ paths in $G - E_i$. This is a contradiction, and therefore Claim 2 is true.

CLAIM 3. There is an index i for which $V_i V_{i+1} V_i^{**} V_{i+1}^{**}$ are in cyclic order on S.

Proof. Let i be an index for which the directed (in the sense of increasing i modulo r) arc from V_i to V_i^{**} on S contains the minimal number of intermediate vertices of S, among all such possible (directed) arcs. This minimal number is not zero, because V_i^{**} is different from V_i and from its two neighbors in S. If V_{i+1}^{**} were between V_{i+1} and V_i^{**}, the arc from V_{i+1} to V_{i+1}^{**} on S would contain less vertices than the arc from V_i to V_i^{**}, which contradicts the choice of i.

Therefore V_{i+1}^{**} is not between V_{i+1} and V_i^{**}, as required.

CLAIM 4. The index i, as given by Claim 3, is such that $G - E_i$ is 2-connected.

Proof. Suppose, on the contrary, that $G - E_i$ is not 2-connected. Since G is 2-connected, it follows, using Lemma 1, that $G - E_i$ is only 1-connected between V_i and V_{i+1}.

To prove that this is not the case, we will show that $G - E_i$ contains two $V_i - V_{i+1}$ paths, disjoint except for common end points, as follows:

If $\alpha_i \cap \alpha_{i+1} \neq \varnothing$, then $S - E_i$ is one such a path, while the other is contained in $\alpha_i \cup \alpha_{i+1} \cup E_i^* \cup E_{i+1}^*$.

If $\alpha_i \cap \alpha_{i+1} = \varnothing$, then $E_i^* \cup \alpha_i$ together with the part of S from V_{i+1} to V_i^{**} is one path, while the other one is $E_{i+1}^* \cup \alpha_{i+1}$, together with that part of S between V_{i+1}^{**} and V_i, which misses V_{i+1}. These two parts of S are disjoint because $V_i V_{i+1} V_i^{**} V_{i+1}^{**}$ are in cyclic order in S, as given by Claim 3.

The proof of Claim 4 is complete.

Clearly, Claims 1 and 4 are in contradiction, obtained by assuming that no vertex of G is totally covered by $F(G)$. The proof of Lemma 4 is complete.

Proof of Theorem 2. Theorem 2 states that, if an n-connected graph has a 1-factor, it has at least as many as $n(n-2)(n-4) \ldots$ 1-factors, $n \geqslant 1$.

[†] $i + 1$, as an index of a vertex of S, is $i + 1 \pmod r$.

Since the assertion is trivial for $n = 1$, while it becomes just Theorem 1 for $n = 2$, we will assume that $n \geqslant 3$.

Every n-connected graph G with a 1-factor has, by Lemma 4, a vertex V which is totally covered by $F(G)$, since G is 2-connected. The valence of V is at least n, therefore the condition of Lemma 3 holds, and the proof of Theorem 2 follows immediately from Lemma 3.

To show that equality holds in $f(n) \geqslant n(n-2)(n-4) \ldots 5 \cdot 3$, for odd n, observe that the complete graph with $n+1$ vertices, for odd n, has precisely $n(n-2)(n-4) \ldots 5 \cdot 3$ 1-factors.

VI. Remarks. We extend our Lemma 4 as follows:

Corollary 1. *If G is a 2-connected graph with a 1-factor, then G contains two vertices which are totally covered by $F(G)$.*

Proof. The proof is by induction on the number k of the edges of G, and is very similar to the proof of Lemma 4.

If $k = 4$, the only 2-connected graph with 1-factor having 4 edges is the cycle of length 4, which has all four of its vertices totally covered by its 1-factors.

Suppose our corollary is true for all the appropriate graphs with $k - 1$ edges and let G be a 2-connected graph with k edges, such that G has a 1-factor F. G has by [1] another 1-factor, say F'.

Suppose now that G has at most one vertex, which is totally covered by $F(G)$.

Claim 1. If G has an edge E, which belongs to one 1-factor F^* of G, but does not belong to another 1-factor F^{**} of G, then $G - E$ is only 1-connected.

Proof. The proof is similar to the proof of Claim 1 in Lemma 4, except that here two vertices of G are under consideration of being totally covered by $F(G)$.

Next, let $S = (V_1, E_1, V_2, E_2, \ldots, E_r, V_1)$ be a closed cycle, such that $E_{2i} \in F$ and $E_{2i+1} \in F'$, as in the proof of Lemma 4, $1 \leqslant 2i, 2i + 1 \leqslant r$.

It follows from Claim 1 here that $G - E_i$ is 1-connected, for all $1 \leqslant i \leqslant r$.

If a vertex V_j of S is 2-valent in G, the two edges of G meeting at V_j are E_{j-1} and E_j, hence V_j is totally covered by $F(G)$. As a consequence of the assumption on G, at most one vertex of S is 2-valent (the rest have clearly valences $\geqslant 3$).

If no vertex of S is 2-valent, the rest of the proof of Lemma 4 is applicable (i.e., construct α_i, $1 \leqslant i \leqslant r$, and prove Claims 2, 3, and 4), thus contradicting the assumption on G.

In case one vertex of S, say V_1, is 2-valent, we proceed as follows: since the valence of $V_i \geqslant 3$, for all $2 \leqslant i \leqslant r$, there are edges $E_2^*,\ldots, E_r^*$ of G which are not edges of S, such that V_i is a vertex of E_i^*, for all $2 \leqslant i \leqslant r$. Let $E_i^* = (V_i, V_i^*)$, $2 \leqslant i \leqslant r$. Since $G - V_i$ is connected, let α_i $(2 \leqslant i \leqslant r)$ be a path in $G - V_i$, connecting V_i^* to some V_i^{**} on S, such that V_i^{**} is the only vertex of S on α_i; if $V_i^* \in S$, then α_i degenerates to the vertex V_i^* and $V_i^{**} = V_1^*$. Clearly, $V_i^{**} \neq V_i$.

Claim 2. V_i^{**} is not a neighbor in S of V_i, for all $2 \leqslant i \leqslant r$.

The proof is as in the corresponding case in Lemma 4.

Claim 3. There is an index i, $2 \leqslant i \leqslant r - 1$, for which $V_i V_{i+1} V_i^{**} V_{i+1}^{**}$ are in cyclic order on S.

Proof. Observe that since V_1 is 2-valent, $V_i^{**} \neq V_1$ for all $2 \leqslant i \leqslant r$.

The directed arc, in the sense of increasing i modulo r, from V_2 to V_2^{**} on S does not contain V_1. Let i, $2 \leqslant i \leqslant r$, be such that the directed arc from V_i to V_i^{**} on S does not contain V_1 and contains the minimal possible number of intermediate vertices.

If V_{i+1}^{**} belongs to the directed arc from V_i to V_i^{**} on S, then the directed arc from V_{i+1} to V_{i+1}^{**} on S does not contain V_1 and has less vertices than the directed arc from V_i to V_i^{**} on S; this contradicts the choice of i.

Therefore V_{i+1}^{**} does not belong to the directed arc from V_i to V_i^{**} on S, and $V_i V_{i+1} V_i^{**} V_{i+1}^{**}$ are in cyclic order on S, as claimed.

Claim 4. The index i, as given by Claim 3, is such that $G - E_i$ is 2-connected.

The proof of this claim is the same as in the corresponding case in Lemma 4; Claims 1 and 4 contradict each other, hence the assumption on G is disproved, and G has two vertices which are totally covered by $F(G)$. The proof of Corollary 1 is complete.

If the valences of the vertices of a graph G are all $\geqslant k$ for $k \geqslant 2$ and V_1 and V_2 are vertices of G, then clearly the valences of the vertices of $G - \{V_1, V_2\}$ are all $\geqslant k - 2$. Therefore, using previous results implies

Corollary 2. *If G is an n-connected graph, $n \geqslant 2$, G has a* 1*-factor and the valences of the vertices of G are all $\geqslant k$ (hence: $k \geqslant n$), then G has at least as many as the following number of* 1*-factors:*

(i) $k(k-2)(k-4)\ldots(k-n+2)$ *for even n, or*

(ii) $k(k-2)(k-4)\ldots(k-n+3)$ *for odd n.*

Theorem 2 gives the precise value of $f(n)$, for all the odd n, while for the even n it gives only a lower bound.

Since every (simple) cycle of even length has precisely two 1-factors, it follows that $f(2) = 2$.

The party graph[‡] P_{2k} is defined as the complement of the graph consisting of k disjoint edges. P_6 is a 4-connected graph with precisely eight 1-factors, hence $f(4) \leqslant 8$, which together with Theorem 2 implies that $f(4) = 8$.

For even n, $n \geqslant 6$, we have the following:

$$48 \leqslant f(6) \leqslant 60,$$
$$384 \leqslant f(8) \leqslant 544,$$
$$3840 \leqslant f(10) \leqslant 6040, \text{ etc.},$$

where the upper bound for $f(n)$ is obtained using the graph P_{n+2}.

VII. CONJECTURES. Since $f(2k)$ was not determined here for $k \geqslant 3$, we would like to raise the following

CONJECTURE 1. $f(2k) > 2^k \cdot k!$, *for all* $k \geqslant 3$.

Relating to Corollary 1, we have

CONJECTURE 2. *If G is an n-connected graph with a 1-factor, then G has n vertices which are totally covered by F(G), for all* $n \geqslant 3$.

In another direction we have

CONJECTURE 3. *For every odd* $n \geqslant 3$ *and for* $n = 4$, *there is a unique graph* G_n *which is n-connected and has precisely* $f(n)$ *1-factors.*

Assuming the last conjecture is true, let $f^*(n)$ be the maximum k for which the following is true: "If G is an n-connected graph with a 1-factor, and $G \neq G_n$, then G has at least k (different) 1-factors."

CONJECTURE 4. $f^*(3) = 4, f^*(4) = 10$ *and* $f^*(5) = 30$, *and* $f^*(n) > f(n)$ *for all odd* $n \geqslant 3$ *and* $n = 4$.

[‡] P_{2k} "represents" a cocktail party of k couples, where everybody is talking to everybody else, except for each husband talking to his wife.

Let $f(n, v)$ be the maximum k for which the following is true: "If G is an n-connected graph with v vertices, and G has a 1-factor, then G has at least k (different) 1-factors." Let $B(n, k)$ denote the graph, obtained from the complete bipartite graph on n and $n + 2k$ vertices, by adding k disjoint edges with end points in the set of the $n + 2k$ vertices. $B(n, k)$ is n-connected and has $n!$ 1-factors, therefore $f(n, v) \leqslant n!$ for all $v \geqslant 2n$.

CONJECTURE 5. *$f(n, v) = n!$, for all $v \geqslant 2n$, $n \geqslant 3$.*

$B(n, 1)$ is a counter example to the following question:† "If G is an n-connected graph with a 1-factor, and n is large enough, does G have two 1-factors which have no edge in common?"

Most of the conjectures here are due to B. Grünbaum.

VIII. MAXIMAL MATCHINGS. A *matching* M of a graph G is a subgraph of G, which contains all the vertices of G, each with valence 0 or 1; M is called *maximal matching* if M is a matching and has the maximum possible number of edges. A vertex V of M with valence 0 in M is called *isolated* (in M).

A vertex V of a graph G is called *totally covered by* $M(G)$ if each edge of G, incident to V, belongs to a maximal matching of G.

Let $g(n)$ be the maximum k for which the following is true: "If G is an n-connected graph with no 1-factor, then G has at least k (different) maximal matchings."

We have the following

LEMMA 5. *If all the vertices of a graph G are of valence $\geqslant k$, and G has no 1-factors, then G has $k + 1$ vertices which are totally covered by $M(G)$.*

In particular, $n + 1$ vertices of an n-connected graph G with no 1-factors are totally covered by $M(G)$.

Lemma 5 implies

THEOREM 3. *For all $n \geqslant 3$, $g(n) \geqslant n \cdot g(n-2) + \min\{g(n-1), f(n-1)\}$, and $g(1) = 2$, $g(2) = 3$.*

Proof of Lemma 5. Let G be a graph, all vertices of which have valence $\geqslant k$, and let M be a maximal matching of G. Since G has no 1-factors, M has an isolated vertex, call it V_0. The valence of V_0 is $\geqslant k$, by assumption, so let $V_1, \ldots, V_t$ be all of the vertices of G such that (V_0, V_i) is an edge of G, for all $1 \leqslant i \leqslant t$, where $t \geqslant k$.

† Asked by W. T. Tutte, in private communication.

If V_i is an isolated vertex of M, for some $1 \leqslant i \leqslant t$, then $M \cup (V_0, V_i)$ is a matching of G, which has more edges than M has; this contradicts the maximality of M.

Therefore no vertex V_i, for $1 \leqslant i \leqslant t$, is isolated in M. Let us assume, without loss of generality, that the notation is such that (V_i, V_{i+1}) is an edge of M, for all *odd* i, $1 \leqslant i \leqslant 2p - 1$, for some integer p; while (V_i, V_i^*) is an edge of M, for all $2p + 1 \leqslant i \leqslant t$, where V_i^* are vertices of G, such that $\{V_i \mid 1 \leqslant i \leqslant t\} \cap \{V_i^* \mid 2p + 1 \leqslant i \leqslant t\} = \varnothing$. Clearly, since the edges of a matching are disjoint, $V_i^* \neq V_j^*$ for all $i \neq j$ and $2p + 1 \leqslant i, j \leqslant t$.

We exhibit t additional matchings M_j of G, for all $1 \leqslant j \leqslant t$, as follows: for odd j and $1 \leqslant j \leqslant 2p - 1$, let $M_j = [M - (V_j, V_{j+1})] \cup (V_0, V_j)$; for even j and $2 \leqslant j \leqslant 2p$, let $M_j = [M - (V_{j-1}, V_j)] \cup (V_0, V_j)$; for all j, $2p + 1 \leqslant j \leqslant t$, let $M_j = [M - (V_j, V_j^*)] \cup (V_0, V_j)$.

Clearly, each M_j is a matching of G, and, since each one has the same number of edges as M has, M_j is a maximal matching of G, for all $1 \leqslant j \leqslant t$.

Since $(V_0, V_j) \in M_j$ for all $1 \leqslant j \leqslant t$, V_0 is totally covered by $M(G)$. We have established the following

LEMMA. *If V is an isolated vertex of a maximal matching M of the graph G under consideration, then V is totally covered by $M(G)$.*

Observing that V_{j+1} is an isolated vertex of M_j for all odd j and $1 \leqslant j \leqslant 2p - 1$; V_{j-1} is an isolated vertex of M_j for all even j and $2 \leqslant j \leqslant 2p$, and V_j^* is an isolated vertex of M_j for all $2p + 1 \leqslant j \leqslant t$, it follows from the last lemma that G has t more vertices which are totally covered by $M(G)$, and the proof of Lemma 5 is completed.

Proof of Theorem 3. It follows immediately from the proof of Lemma 5 that, if G is n-connected and has no 1-factors, then G has $n + 1$ different maximal matchings, hence $g(n) \geqslant n + 1$.

The graph consisting of two edges with a common vertex is 1-connected and has two maximal matchings, hence $g(1) = 2$; the cycle of length 3 is 2-connected and has three maximal matchings, hence $g(2) = 3$.

To prove the inequality in question, let $n \geqslant 3$ and let G be an n-connected graph with no 1-factors. It follows by Lemma 5 that G has a vertex V_0 totally covered by $M(G)$. Let (V_0, V_i), for all $1 \leqslant i \leqslant t$, be all the edges of G, incident to V_0. Since G is n-connected, $t \geqslant n$.

The graph $G - \{V_0, V_i\}$, for all $1 \leqslant i \leqslant t$, is $(n - 2)$-connected and has no 1-factors, since adding the edge (V_0, V_i) to such a 1-factor yields a 1-factor of G, contradicting the assumption on G. However, if M^i is a maximal matching of $G - \{V_0, V_i\}$, $M^i \cup (V_0, V_i)$ is a maximal matching of G, for all $1 \leqslant i \leqslant t$, and $M^i \neq M^j$ for $i \neq j$. Therefore G has at least $t \cdot g(n-2)$ maximal matchings, each one of which contains an edge of the form (V_0, V_i), for a suitable i, $1 \leqslant i \leqslant t$.

In addition, G has maximal matchings having V_0 as an isolated vertex, as follows: $G - V_0$ is $(n-1)$-connected and has either $f(n-1)$ 1-factors, or $g(n-1)$ maximal matchings (if it has no 1-factors). If $\overline{M}$ is 1-factor of $G - V_0$, then $\overline{M} \cup \{V_0\}$ is a maximal matching of G of the required kind. If $\overline{M}$ is a maximal matching of $G - V_0$ but $\overline{M} \cup V_0$ is not a maximal matching of G, then a maximal matching of G has one more edge than a maximal matching of $G - V_0$. However this contradicts the existence of a maximal matching of G having V_0 as an isolated vertex. Therefore $\overline{M} \cup V_0$ is a maximal matching of G of the required kind, hence G has at least $\min\{g(n-1), f(n-1)\}$ maximal matching with V_0 as an isolated vertex, and the proof of Theorem 3 is completed.

Except for the obvious questions that can be raised for the values of $g(n)$, $n \geqslant 3$, we state the following

CONJECTURE 6. *For all $n \geqslant 3$, $g(n) > f(n)$.*

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Added in proof: Conjecture 2 has been proved by L. Lovász (private communication).

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